

Integrable discrete systems on \mathbb{R} and related dispersionless systems

Maciej Błaszak¹, Metin Gürses², Burcu Silindir²
and Błażej M. Szablikowski¹

(1) Institute of Physics, A. Mickiewicz University
Umultowska 85, 61-614 Poznań, Poland

E-mails: blaszakm@amu.edu.pl and bszablik@amu.edu.pl

(2) Department of Mathematics, Faculty of Sciences
Bilkent University, 06800 Ankara, Turkey

E-mails: gurses@fen.bilkent.edu.tr and silindir@fen.bilkent.edu.tr

Abstract

The general framework for integrable discrete systems on \mathbb{R} in particular containing lattice soliton systems and their q -deformed analogues is presented. The concept of regular grain structures on \mathbb{R} , generated by discrete one-parameter groups of diffeomorphisms, through which one can define algebras of shift operators is introduced. Two integrable hierarchies of discrete chains together with bi-Hamiltonian structures are constructed. Their continuous limit and the inverse problem based on the deformation quantization scheme are considered.

1 Introduction

In recent years of wide interest have become the so-called integrable q -analogues of KP and Toda types hierarchies together with related Hamiltonian structures, W -algebras, τ -functions, etc. (see [1]-[9] and references therein). The q -deformed KP hierarchy (q -KP) with the reductions of q -KdV soliton type systems are obtained by means of pseudo-differential operators defined through q -derivative ∂_q instead of the usual derivative ∂ used for ordinary KP and KdV hierarchies:

$$\partial u(x) = \frac{\partial u(x)}{\partial x} \quad \rightarrow \quad \partial_q u(x) = \frac{u(qx) - u(x)}{(q-1)x}.$$

Analogously q -deformed Toda hierarchies can be constructed by means of q -shift operators:

$$Eu(x) = u(x+1) \quad \rightarrow \quad E_q u(x) = u(qx).$$

The scheme of the construction of integrable q -deformed systems is based on the classical R -matrix formalism that proved very fruitful for systematic construction of field and lattice

soliton systems [10]-[15] as well as dispersionless integrable field systems [16]-[19]. Moreover, the R -matrix approach allows a construction of Hamiltonian structures and conserved quantities. By integrable systems we understand these which have infinite hierarchy of symmetries and conserved quantities.

Having all the above classes of integrable systems, with parallel schemes of construction, it is interesting how to embed them into a more general unifying framework. One way of doing this is a construction of integrable systems on time scales [20, 21]. A time scale \mathbb{T} is an arbitrary nonempty closed subset of real numbers. It was introduced to unify all possible intervals on the real line \mathbb{R} , like continuous (whole) \mathbb{R} , discrete \mathbb{Z} , and q -discrete \mathbb{K}_q intervals. On a given time scale it is possible to construct Δ -derivative (being simultaneously generalization of the ordinary derivative and the q -derivative) through forward $\sigma(x)$ and backward $\rho(x)$ jump operators, where $x \in \mathbb{T}$ (for all precise definitions see [20, 21]). Assuming the regularity property of \mathbb{T} , implying that $\rho(\sigma(x)) = x$, one can define an algebra of Laurent series of Δ -operators

$$\Delta u(x) = \frac{u(\sigma(x)) - u(x)}{\mu(x)} \quad \mu(x) \equiv \sigma(x) - x, \quad x \in \mathbb{T}$$

or shift operators $E u(x) = u(\sigma(x))$, leading to the construction of integrable systems on time scales. Defining suitable inner products in this algebras, one can construct additionally conservation laws. In such a formulation, dynamical fields $u : \mathbb{T} \rightarrow \mathbb{R}$ are mappings from a time scale to real numbers.

The main goal of this work is the formulation of a general unifying framework of integrable discrete systems, in such a way that the domain of dynamical fields u is always \mathbb{R} . We also consider the continuous limit and the inverse procedure. In the second section we introduce the concept of regular grain structure on \mathbb{R} defined by discrete one-parameter groups of diffeomorphisms $\sigma_{m\hbar}(x)$. Then, the shift operator can be constructed through formal jump operator $\sigma(x) = \sigma_{\hbar}(x)$. In this section elements of geometric scheme are defined as appropriate functionals, duality maps, adjoint operators, etc. A class of discrete systems is chosen in such a way that the limit $\hbar \rightarrow 0$ is dispersionless. In the third section, using the formalism of classical R -matrices, we construct two integrable hierarchies of discrete chains being counterparts of the original infinite-field Toda and modified Toda chains. Additionally bi-Hamiltonian structures are constructed. In the next section the concept of the continuous limit, which in our case becomes the dispersionless limit, is explained. Further, in the fifth section, the theory of dispersionless chains, being dispersionless limits of discrete chains together with bi-Hamiltonian structures is presented. In the sixth section the inverse problem to the dispersionless limit is considered. It is based on the scheme of deformation quantization formalism introduced in [15]. As a result we show that there is a class of gauge equivalent integrable discrete systems being dispersive counterparts of dispersionless systems considered earlier. We end the paper with some final comments.

2 One-parameter regular grain structures on \mathbb{R}

The main aim of this article is the formulation of a general theory of integrable discrete systems on \mathbb{R} , that will contain lattice soliton systems as well as q -discrete systems as particular cases. This theory will be illustrated by integrable discrete chains being infinite-field systems.

Maps $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ and $\rho : \mathbb{R} \rightarrow \mathbb{R}$ will be called the forward and backward jump operators, respectively. These names are only a convention as we do not assume that $\sigma(x) \geq x$ and

$\rho(x) \leq x$ for arbitrary $x \in \mathbb{R}$. In $n \in \mathbb{Z}_+$ forward steps the point $x \in \mathbb{R}$ is mapped to the point $\sigma^n(x)$, where σ^n is the n -times composition of forward jump operator σ . Respectively, in n backward steps, x is mapped to the point $\rho^n(x)$. Then, the range of possible points to which we can map x by forward and backward steps (including x) is given by

$$\mathbb{G}_x := \{\rho^n(x) : n \in \mathbb{Z}_+\} \cup \{x\} \cup \{\sigma^n(x) : n \in \mathbb{Z}_+\}.$$

Hence, to each point x of \mathbb{R} a set \mathbb{G}_x is associated. The union of all \mathbb{G}_x is given by $\mathbb{G} := \bigcup_{x \in \mathbb{R}} \mathbb{G}_x$.

Definition 2.1 *We will say that \mathbb{G} defines the grain structure on \mathbb{R} . We will call it the regular grain structure, if there exist inverse maps σ^{-1} and ρ^{-1} , such that $\sigma(x) = \rho^{-1}(x)$ and $\rho(x) = \sigma^{-1}(x)$ for all $x \in \mathbb{R}$.*

So, to define the regular grain structure on \mathbb{R} one needs only one forward jump operator σ being bijection, as the backward operator is given by σ^{-1} . Then, $\mathbb{G}_x = \{\sigma^n(x) : n \in \mathbb{Z}\}$, where we assumed that $\sigma^0 \equiv \text{id}_{\mathbb{R}}$. Besides, bijective σ defines a discrete one-parameter group of bijections on \mathbb{R} : $\mathbb{Z} \ni m \mapsto \{\sigma_m : \mathbb{R} \rightarrow \mathbb{R}\}$, such that $\sigma_m := \sigma^m$, and vice versa each one-parameter group of bijections on \mathbb{R} defines the regular grain structure on \mathbb{R} with the forward jump operator defined by $\sigma := \sigma_1$. Notice that the regular grain structure introduces equivalence classes between points of \mathbb{R} , such that $x \sim y$ if $\mathbb{G}_x = \mathbb{G}_y$ ($x, y \in \mathbb{R}$), i.e. there exists $k \in \mathbb{Z}$ such that $y = \sigma^k(x)$.

Further on, we introduce a regular grain structure \mathbb{G} on \mathbb{R} by one-parameter group of diffeomorphisms instead of bijections, which is necessary as we will deal with differential geometry of infinite-dimensional systems with smooth dynamical fields. Let $\mathbb{Z} \ni m \mapsto \sigma_{m\hbar}$ be a discrete one-parameter group of diffeomorphisms on \mathbb{R} : $\sigma_{m\hbar} : \mathbb{R} \rightarrow \mathbb{R}$, i.e

$$\sigma_0(x) = x \quad \text{and} \quad \sigma_{m\hbar}(\sigma_{n\hbar}(x)) = \sigma_{(m+n)\hbar}(x) \quad m, n \in \mathbb{Z},$$

where $\hbar > 0$ is some deformation parameter. It follows that $(\sigma_{n\hbar})^{-1}(x) = \sigma_{-n\hbar}(x)$. The continuous one-parameter group of diffeomorphisms ($\mathbb{R} \ni t \mapsto \sigma_t$) can be completely determined by its infinitesimal generator $\mathcal{X}(x)\partial_x$ being a vector field on \mathbb{R} . We assume that the component $\mathcal{X}(x)$ is smooth on \mathbb{R} except at most at a finite number of points. Then,

$$\mathcal{X}(x) = \left. \frac{d\sigma_t(x)}{dt} \right|_{t=0} \Leftrightarrow \frac{d\sigma_t(x)}{dt} = \mathcal{X}(\sigma_t(x)), \quad (2.1)$$

where $t \in \mathbb{R}$. Arbitrary $\mathcal{X}\partial_x$ generates a continuous one-parameter group of diffeomorphisms only when it is a *complete* vector field, for which maximal integrals are defined on the whole \mathbb{R} , i.e. \mathbb{R} is a domain of the mapping $t \mapsto \sigma_t$. In such a case the above discrete one-parameter group is well defined as it is enough to consider subgroup \mathbb{Z} of \mathbb{R} . Incomplete $\mathcal{X}\partial_x$ might still well define a discrete group of diffeomorphisms, if \hbar is properly chosen.

Lemma 2.2 *Let $\sigma_t(x)$ be a one-parameter group of diffeomorphisms generated by $\mathcal{X}(x)\partial_x$. Then, the following relation is valid*

$$\mathcal{X}(x) \frac{d\sigma_t(x)}{dx} = \mathcal{X}(\sigma_t(x)). \quad (2.2)$$

Proof. From (2.1) one observes that $\mathcal{X}(\sigma_{s+t}(x)) = \frac{d\sigma_{s+t}(x)}{ds}$. Hence, the following relation is valid

$$\mathcal{X}(\sigma_s(x)) \frac{d\sigma_{s+t}(x)}{d\sigma_s(x)} = \mathcal{X}(\sigma_{s+t}(x)).$$

Since it can be obtained from (2.2) by acting on its both sides with σ_s , the proposition is completed. \square

Now, we establish a phase space related to discrete systems under considerations. Let

$$\mathbf{u} := (u_0(x), u_1(x), u_2(x), \dots)^T$$

be a infinite-tuple of smooth functions $u_i : \mathbb{R} \rightarrow \mathbb{K}$, $x \mapsto u_i(x)$ with values in $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Additionally we assume that u_i depend on an appropriate set of evolution parameters, and so u_i are dynamical fields. Let \mathcal{U} be a linear topological space, with local independent coordinates $\mathbf{u}(\sigma_{m\hbar}(x))$ for all $m \in \mathbb{Z}$, defining our infinite-dimensional phase space. We will use the following notation

$$(E^m u)(x) \equiv u(\sigma_{m\hbar}(x)).$$

Let \mathcal{C} be the algebra over \mathbb{K} of functions on \mathcal{U} of the form

$$f[\mathbf{u}] := \sum_{m \geq 0} \sum_{i_1, \dots, i_m \geq 0} \sum_{s_1, \dots, s_m \in \mathbb{Z}} a_{s_1 s_2 \dots s_m}^{i_1 i_2 \dots i_m} (E^{s_1} u_{i_1}) (E^{s_2} u_{i_2}) \dots (E^{s_m} u_{i_m}) \quad (2.3)$$

that are polynomials in $\mathbf{u}(\sigma_{m\hbar}(x))$ of finite order, with coefficients $a_{s_1 s_2 \dots s_m}^{i_1 i_2 \dots i_m} \in \mathbb{K}$. This algebra can be extended into operator algebra $\mathcal{C}[E, E^{-1}]$ ($\mathcal{C}[x, y, \dots]$ stands for the linear space of polynomials in x, y, \dots with coefficients from \mathcal{C}), where E is a shift operator compatible with the grain structure defined by $\sigma_{\hbar}(x)$, i.e.

$$E^m u(x) := (E^m u)(x) = u(\sigma_{m\hbar}(x)) \quad m \in \mathbb{Z},$$

where $u(x)$ is some field. As $\sigma_{\hbar}(x)$ is an element of one-parameter group of diffeomorphisms, hence

$$\sigma_{\hbar}(x) = e^{\hbar \mathcal{X}(x) \partial_x} x \quad \Leftrightarrow \quad e^{\hbar \mathcal{X}(x) \partial_x} u(x) = u(e^{\hbar \mathcal{X}(x) \partial_x} x) \quad (2.4)$$

is valid if $\mathcal{X}(x) \partial_x$ is a complete vector field and $u(x)$ is a smooth function. Thus, the shift operator E can be identified with $e^{\hbar \mathcal{X}(x) \partial_x}$, i.e.

$$E^m \equiv e^{m \hbar \mathcal{X}(x) \partial_x}. \quad (2.5)$$

Example 2.3 Consider vector fields on \mathbb{R} of the form $\mathcal{X}(x) \partial_x = x^{1-n} \partial_x$ for $n \in \mathbb{Z}$. For $n = 0$ integrating (2.1) one finds that

$$\sigma_t(x) = e^t x \quad \Rightarrow \quad \sigma_{m\hbar}(x) = e^{m\hbar} x = q^m x \quad q \equiv e^{\hbar},$$

which is defined for all $t \in \mathbb{R}$ and so $\mathcal{X} \partial_x = x \partial_x$ is a complete vector field. When $n = 0$, we will deal with systems of 'q-discrete' type. When $n \neq 0$, in general, $\sigma_t(x)$ can be find only in the implicit form

$$(\sigma_t(x))^n = x^n + nt.$$

For $n = 1$ we have

$$\sigma_t(x) = x + t \quad \Rightarrow \quad \sigma_{m\hbar}(x) = x + m\hbar,$$

and $\mathcal{X}\partial_x = \partial_x$ is obviously complete. In this case we will deal with systems of 'lattice' type. For $n = -1$ the related vector field $\mathcal{X}\partial_x = x^2\partial_x$ is incomplete as $t \neq \frac{1}{x}$:

$$\sigma_t(x) = \frac{x}{1 - tx} \quad \Rightarrow \quad \sigma_{m\hbar}(x) = \frac{x}{1 - m\hbar x}.$$

However, if $x \neq \frac{1}{m\hbar}$, the related discrete one-parameter group of diffeomorphisms is well defined. When n is odd, we can always define a discrete one-parameter group of diffeomorphisms generated by $\mathcal{X}\partial_x = x^{1-n}\partial_x$. Another example of incomplete vector field is given for $n = 2$ ($\mathcal{X}\partial_x = \frac{1}{x}\partial_x$) since we have

$$\sigma_t(x) = \text{sign}(x)\sqrt{x^2 + 2t}$$

with restriction $t \geq -\frac{1}{2}x^2$. The vector fields $\mathcal{X}\partial_x = x^{1-n}\partial_x$ of this kind should be excluded from further considerations or properly extended over complex plane \mathbb{C} .

A space $\mathcal{F} = \{F : \mathcal{U} \rightarrow \mathbb{K}\}$ of functions on \mathcal{U} is defined through linear functionals

$$\int (\cdot) d_{\hbar}x : \mathcal{C} \rightarrow \mathbb{K} \quad f[\mathbf{u}] \mapsto F(\mathbf{u}) := \int f[\mathbf{u}] d_{\hbar}x, \quad (2.6)$$

such that

$$\int Ef[\mathbf{u}] d_{\hbar}x = \int f[\mathbf{u}] d_{\hbar}x, \quad (2.7)$$

where $\int d_{\hbar}x$ is a formal integration symbol. The restriction (2.7) entails the form of adjoint with respect to the duality map which will be defined in a moment.

Definition 2.4 *The explicit form of appropriate functionals can be introduced in two ways.*

(i) *A discrete representation is defined as*

$$F(\mathbf{u}) = \int f[\mathbf{u}] d_{\hbar}x := \hbar \sum_{n \in \mathbb{Z}} f[\mathbf{u}(\sigma_{n\hbar}(x))]. \quad (2.8)$$

(ii) *A continuous representation is given as*

$$F(\mathbf{u}) = \int f[\mathbf{u}] d_{\hbar}x := \int_{-\infty}^{\infty} f[\mathbf{u}(x)] \frac{dx}{\mathcal{X}(x)}, \quad (2.9)$$

where we assume that $u_i(x)$ vanishes as $|x| \rightarrow \infty$ (if $\mathcal{X}(x) \rightarrow 0$ for $|x| \rightarrow \infty$, then $u_i(x)$ must vanish faster than $\mathcal{X}(x)$ does). The above integral in general is improper, so additionally we have to assume that $u_i(x)$ behave properly as x tends to critical points x_c of $\mathcal{X}(x)$ ($\mathcal{X}(x_c) = 0$). Then, evaluating the integral we take its principal value.

When it is not necessary to differentiate between the above representations, we will use only the formal integration symbol $\int d_{\hbar}x$. We have explicitly defined the functionals in two ways reflecting two different approaches developed for the lattice soliton systems. The first one is with the domain of dynamical fields \mathbb{Z} [12, 13], the second one with \mathbb{R} [15, 22]. So, the functionals (2.8) and (2.9) are appropriate generalizations of these two approaches.

Proposition 2.5 *Both functionals from Definition 2.4 are well defined and satisfy (2.7).*

Proof. Both functionals are trivially linear. The discrete one satisfies (2.7) since we can freely change the boundaries because we sum over the whole \mathbb{Z} . For the second functional we have that

$$\begin{aligned} \int E f[\mathbf{u}] d_{\hbar} x &= \int_{-\infty}^{\infty} f[\mathbf{u}(\sigma_{\hbar}(x))] \frac{dx}{\mathcal{X}(x)} = \int_{-\infty}^{\infty} f[\mathbf{u}(x)] \frac{d\sigma_{-\hbar}(x)}{dx} \frac{dx}{\mathcal{X}(\sigma_{-\hbar}(x))} \\ &= \int_{-\infty}^{\infty} f[\mathbf{u}(x)] \frac{dx}{\mathcal{X}(x)} = \int f[\mathbf{u}] d_{\hbar} x, \end{aligned}$$

where one obtains the second equality by the change of variables $x \mapsto \sigma_{\hbar}(x)$, while the next one follows from Lemma 2.2. \square

A vector field on \mathcal{U} is given by a system of differential-difference equations, the difference one with respect to the grain structure defined by σ_{\hbar} and the first order differential one with respect to the evolution parameter t ,

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}), \quad (2.10)$$

where $u_t := \frac{\partial u}{\partial t}$ and $\mathbf{K}(\mathbf{u}) := \frac{1}{\hbar}(K_1[\mathbf{u}], K_2[\mathbf{u}], \dots)^T$ with $K_i[\mathbf{u}] \in \mathcal{C}$. The class of the discrete systems is chosen in such the way that in the continuous limit $\hbar \rightarrow 0$ we obtain systems of hydrodynamic type (see Section 4). This assumption explains the appearance of the factor containing \hbar in \mathbf{K} .

Let \mathcal{V} be a linear space over \mathbb{K} , of all such vector fields on \mathcal{U} . Then the dual space \mathcal{V}^* is a space of all linear maps $\eta : \mathcal{V} \rightarrow \mathbb{K}$. The action of $\eta \in \mathcal{V}^*$ on $\mathbf{K} \in \mathcal{V}$ can be defined through a duality map (bilinear functional) $\langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \rightarrow \mathbb{K}$ given by functional (2.6) as

$$\langle \eta, \mathbf{K} \rangle = \int \sum_{i=0}^{\infty} \eta_i K_i d_{\hbar} x = \int \eta^T \cdot \mathbf{K} d_{\hbar} x, \quad (2.11)$$

where components of $\eta := (\eta_1, \eta_2, \dots)^T$ belong to \mathcal{C} . With respect to the duality map (2.11) one finds that the adjoint of E^m is equal to E^{-m} , i.e. $(E^m)^\dagger = E^{-m}$.

Proposition 2.6 *The differential*

$$dF(\mathbf{u}) = \left(\frac{\delta F}{\delta u_0}, \frac{\delta F}{\delta u_1}, \dots \right)^T \in \mathcal{V}^*$$

of a functional $F(\mathbf{u}) = \int f[\mathbf{u}] d_{\hbar} x$, such that its pairing with $\mathbf{K} \in \mathcal{V}$ assumes the usual Euclidean form

$$F'[\mathbf{K}] = \langle dF, \mathbf{K} \rangle = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i} (u_i)_t d_{\hbar} x, \quad (2.12)$$

where $F'[\mathbf{K}]$ is the directional derivative, is defined by variational derivatives of the form

$$\frac{\delta F}{\delta u_i} := \sum_{m \in \mathbb{Z}} E^{-m} \frac{\partial f[\mathbf{u}]}{\partial u_i(\sigma_{m\hbar}(x))}.$$

Proof. Let $\mathbf{u}_t = \mathbf{K}(\mathbf{u})$, then

$$F'(\mathbf{u})[\mathbf{u}_t] \equiv \frac{dF(\mathbf{u})}{dt} = \int \sum_{i=0}^{\infty} \sum_{m \in \mathbb{Z}} \frac{\partial f[\mathbf{u}]}{\partial u_i(\sigma_{m\hbar}(x))} \frac{du_i(\sigma_{m\hbar}(x))}{dt} d_{\hbar}x = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i}(u_i)_t d_{\hbar}x,$$

where the last equality follows from (2.7). \square

Further we will be interested in bi-vector fields on \mathcal{U} defined through linear operators $\pi : \mathcal{V}^* \rightarrow \mathcal{V}$, which in a local representation are matrices with coefficients from $\mathcal{C}[E, E^{-1}]$ multiplied by $\frac{1}{\hbar}$. An operator π is a Poisson operator (tensor) if the bilinear bracket

$$\{H, F\}_{\pi} = \langle dF, \pi dH \rangle \quad F, H \in \mathcal{F}$$

is a Poisson bracket.

Remark 2.7 *It is important to mention that the particular choice of the algebra \mathcal{C} , and consequently the algebra $\mathcal{C}[E, E^{-1}]$, determines the class of discrete systems considered, which in the limit $\hbar \rightarrow 0$ tend to differential systems of first order, i.e. dispersionless ones. Alternative approach to the construction of discrete systems on \mathbb{R} with the grain structure \mathbb{G} is based on the use of Δ -derivative, instead of the shift operation, given by*

$$\Delta u(x) := \frac{(E-1)u(x)}{(E-1)x} = \frac{u(\sigma_{\hbar}(x)) - u(x)}{\mu_{\hbar}(x)} \quad \mu_{\hbar}(x) \equiv \sigma_{\hbar}(x) - x.$$

In this case, the algebra \mathcal{C} is composed of polynomials in $\Delta^m u$ ($m = 0, 1, \dots$) and the operator algebra is given by $\mathcal{C}[\Delta]$. Consequently the restriction (2.7) on the functional is replaced by

$$\int' \Delta f[u] d_{\hbar}x = 0, \quad (2.13)$$

which entails that $\Delta^{\dagger} = \Delta E^{-1}$ with respect to the duality map generated by this functional. Prime in \int' is used to differ the functional satisfying property (2.13) from the functional satisfying property (2.7). Nevertheless, both functionals are interrelated by the relation

$$\int' (\cdot) d_{\hbar}x = \int (\cdot) \mu_{\hbar}(x) d_{\hbar}x,$$

which is a consequence of the restrictions imposed on them. Contrary to the previous case the continuous limit of discrete systems from the alternative approach with Δ -operation gives dynamical field systems with dispersion and is not considered in this article.

3 R -matrix approach to integrable discrete systems on \mathbb{R}

Now, we are ready for the construction of integrable discrete systems following from the scheme of classical R -matrix formalism parallel to the one used in the case of lattice soliton systems [12, 14, 15].

On \mathbb{R} with the grain structure \mathbb{G} defined by some diffeomorphism σ_{\hbar} we introduce the algebra of shift operators with finite highest order:

$$\mathfrak{g} = \mathfrak{g}_{\geq k-1} \oplus \mathfrak{g}_{< k-1} = \left\{ \sum_{i \geq k-1}^N u_i(x) \mathcal{E}^i \right\} \oplus \left\{ \sum_{i < k-1} u_i(x) \mathcal{E}^i \right\} \quad (3.1)$$

where

$$\mathcal{E}^m u(x) = (E^m u)(x) \mathcal{E}^m \equiv u(\sigma_{m\hbar}(x)) \mathcal{E}^m \quad \sigma_{m\hbar} := \sigma_{\hbar}^m \quad m \in \mathbb{Z} \quad (3.2)$$

and $u_i(x)$ are smooth dynamical fields.

Proposition 3.1 *The multiplication operation in \mathfrak{g} defined by (3.2) is non-commutative and associative.*

Proof. Non-commutativity is obvious. Associativity follows from straightforward calculation and from the fact that $\sigma_{m\hbar}$ is a one-parameter group of diffeomorphisms. \square

The Lie structure on \mathfrak{g} is introduced through the commutator

$$[A, B] = \frac{1}{\hbar} (AB - BA) \quad A, B \in \mathfrak{g}.$$

The subsets $\mathfrak{g}_{\geq k-1}$ and $\mathfrak{g}_{< k-1}$ of \mathfrak{g} are Lie subalgebras only for $k = 1$ and $k = 2$. As a result, we can define the classical R -matrices $R = P_{\geq k-1} - \frac{1}{2}$, through appropriate projections, and related Lax hierarchies:

$$L_{t_n} = [(L^n)_{\geq k-1}, L] = \pi_0 dH_n = \pi_1 dH_{n-1} \quad n \in \mathbb{Z}_+ \quad k = 1, 2, \quad (3.3)$$

of infinitely many mutually commuting systems. The evolution equations from (3.3) are generated by powers of appropriate Lax operators $L \in \mathfrak{g}$ of the form:

$$k = 1 : \quad L = \mathcal{E} + u_0 + u_1 \mathcal{E}^{-1} + u_2 \mathcal{E}^{-2} + \dots = \mathcal{E} + \sum_{i \geq 0} u_i \mathcal{E}^{-i} \quad (3.4)$$

$$k = 2 : \quad L = u_0 \mathcal{E} + u_1 + u_2 \mathcal{E}^{-1} + u_3 \mathcal{E}^{-2} + \dots = \sum_{i \geq 0} u_i \mathcal{E}^{1-i}. \quad (3.5)$$

Then, the first chains from (3.3) are:

$$\begin{aligned} (u_i)_{t_1} &= \frac{1}{\hbar} [(E - 1)u_{i+1} + u_i(1 - E^{-i})u_0] \\ (u_i)_{t_2} &= \frac{1}{\hbar} [(E^2 - 1)u_{i+2} + Eu_{i+1}(E + 1)u_0 - u_{i+1}(E^{-i} + E^{-i-1})u_0 \\ &\quad + u_i(1 - E^{-i})u_0^2 + u_i(E + 1)(1 - E^{-i})u_1] \\ &\vdots \end{aligned} \quad (3.6)$$

for $k = 1$, and

$$\begin{aligned} (u_i)_{t_1} &= \frac{1}{\hbar} [u_0 E u_{i+1} - u_{i+1} E^{-i} u_0] \\ (u_i)_{t_2} &= \frac{1}{\hbar} [u_0 E u_0 E^2 u_{i+2} - u_{i+2} E^{-i-1} u_0 E^{-i} u_0 \\ &\quad + u_0 (E + 1) u_1 E u_{i+1} - u_{i+1} E^{-i} u_0 (E^{1-i} + E^{-i}) u_1] \\ &\vdots \end{aligned}$$

for $k = 2$. Here and further on the shift operators E^m in evolution equations and conserved quantities act only on the nearest field on the right and in Poisson operators act on everything on the right of the symbol E^m inside and outside the operator.

Example 3.2 *The lattice case: $\mathcal{X} = 1$.*

Let $\hbar = 1$. The first chains of evolution equations from (3.3) have the form:

$$\begin{aligned} k = 1 : \quad & u_i(x)_{t_1} = u_{i+1}(x+1) - u_{i+1}(x) + u_i(x) (u_0(x) - u_0(x-i)) \\ k = 2 : \quad & u_i(x)_{t_1} = u_0(x)u_{i+1}(x+1) - u_0(x-i)u_{i+1}(x). \end{aligned}$$

These are Toda and modified Toda chains, respectively.

Example 3.3 *The q -discrete case: $\mathcal{X} = x$ ($q \equiv e^{\hbar}$).*

In this case the same evolution equations are

$$\begin{aligned} k = 1 : \quad & u_i(x)_{t_1} = u_{i+1}(qx) - u_{i+1}(x) + u_i(x) (u_0(x) - u_0(q^{-i}x)) \\ k = 2 : \quad & u_i(x)_{t_1} = u_0(x)u_{i+1}(qx) - u_0(q^{-i}x)u_{i+1}(x), \end{aligned}$$

where the constant factor \hbar is absorbed into the evolution parameter t_1 through simple rescaling. These are q -deformed analogues of the chains from the previous example.

In this work we do not consider finite-field reductions of (3.3) as the procedure is straightforward following [12, 15].

To construct Hamiltonian structures for (3.3) at first one has to define an appropriate inner product on \mathfrak{g} .

Definition 3.4 *Let $\text{Tr} : \mathfrak{g} \rightarrow \mathbb{K}$ be a trace form, being a linear map, such that*

$$\text{Tr}(A) := \int \text{res}(A\mathcal{E}^{-1}) d_{\hbar}x,$$

where $\text{res}(A\mathcal{E}^{-1}) := a_0$ for $A = \sum_i a_i \mathcal{E}^i$. Then, the bilinear map $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ defined as

$$(A, B) := \text{Tr}(AB) \tag{3.7}$$

is an inner product on \mathfrak{g} .

Proposition 3.5 *The inner product (3.7) is nondegenerate, symmetric and ad-invariant, i.e.*

$$([A, B], C) = (A, [B, C]) \quad A, B, C \in \mathfrak{g}.$$

Proof. The nondegeneracy of (3.7) is obvious. The symmetricity follows from (2.7). The ad-invariance is a consequence of the associativity of multiplication operation in \mathfrak{g} . \square

Next, the differentials $dH(L)$ of functionals $H(L) \in \mathcal{F}(\mathfrak{g})$ for (3.4-3.5) have the form:

$$\begin{aligned} k = 1 : \quad & dH = \sum_{i \geq 0} \mathcal{E}^i \frac{\delta H}{\delta u_i} \\ k = 2 : \quad & dH = \sum_{i \geq 0} \mathcal{E}^{i-1} \frac{\delta H}{\delta u_i}, \end{aligned}$$

which follows from the assumption that inner product on \mathfrak{g} is compatible with (2.12), i.e.

$$(dH, L_t) = \int \sum_{i=0}^{\infty} \frac{\delta H}{\delta u_i} (u_i)_t d_{\hbar}x.$$

Then, the bi-Hamiltonian structure of the Lax hierarchies (3.3) is defined through the compatible (for fixed k) Poisson tensors given by

$$k = 1, 2 : \quad \pi_0 : dH \mapsto [L, (dH)_{<k-1}] + ([dH, L])_{<2-k}$$

and

$$\begin{aligned} k = 1 : \quad \pi_1 : dH &\mapsto \frac{1}{2} \left([L, (LdH + dHL)_{<0}] + L([dH, L])_{<1} + ([dH, L])_{<1} L \right) \\ &\quad + \hbar [(E+1)(E-1)^{-1} \text{res}([dH, L] \mathcal{E}^{-1}), L] \\ k = 2 : \quad \pi_1 : dH &\mapsto \frac{1}{2} \left([L, (LdH + dHL)_{<1}] + L([dH, L])_{<0} + ([dH, L])_{<0} L \right), \end{aligned}$$

where the operation $(E-1)^{-1}$ must be understood formally as the inverse of $(E-1)$ and one can show that $(E+1)(E-1)^{-1} = \sum_{i=1}^{\infty} (E^{-i} - E^i)$. The appropriate Hamiltonians, being conserved quantities, are

$$H_n(L) = \frac{1}{n+1} \text{Tr}(L^{n+1}) \quad dH_n(L) = L^n$$

and the explicit bi-Hamiltonian structure of (3.3) is given by

$$(u_i)_{t_n} = \sum_{j \geq 0} \pi_0^{ij} \frac{\delta H_n}{\delta u_j} = \sum_{j \geq 0} \pi_1^{ij} \frac{\delta H_{n-1}}{\delta u_j} \quad i \geq 0.$$

The Poisson tensors for $k = 1$ are

$$\begin{aligned} \pi_0^{ij} &= \frac{1}{\hbar} [E^j u_{i+j} - u_{i+j} E^{-i}] \\ \pi_1^{ij} &= \frac{1}{\hbar} \left[\sum_{k=0}^i (u_k E^{j-k} u_{i+j-k} - u_{i+j-k} E^{k-i} u_k + u_i (E^{j-k} - E^{-k}) u_j) \right. \\ &\quad \left. + u_i (1 - E^{j-i}) u_j + E^{j+1} u_{i+j+1} - u_{i+j+1} E^{-i-1} \right] \end{aligned}$$

together with the hierarchy of Hamiltonians in the form

$$\begin{aligned} H_0 &= \int u_0 d_{\hbar} x \\ H_1 &= \int \left(u_1 + \frac{1}{2} u_0^2 \right) d_{\hbar} x \\ H_2 &= \int \left(u_2 + u_0(E+1)u_1 + \frac{1}{3} u_0^3 \right) d_{\hbar} x \\ &\vdots \end{aligned}$$

For $k = 2$ the first Poisson tensor has the following form

$$\begin{aligned} \pi_0^{10} &= \frac{1}{\hbar} (1 - E^{-1}) u_0 & \pi_0^{01} &= \frac{1}{\hbar} u_0 (E - 1) \\ \pi_0^{ij} &= \frac{1}{\hbar} [E^{j-1} u_{i+j-1} - u_{i+j-1} E^{1-i}] & i, j &\geq 2, \end{aligned}$$

with all remaining π_0^{ij} equal zero, the second one is

$$\pi_1^{ij} = \frac{1}{\hbar} \left[\sum_{k=0}^{i-1} (u_k E^{j-k} u_{i+j-k} - u_{i+j-k} E^{k-i} u_k) + \frac{1}{2} u_i (E^{1-i} - 1)(E^{j-1} + 1) u_j \right]$$

and the first Hamiltonians are

$$\begin{aligned} H_0 &= \int u_1 d_{\hbar} x \\ H_1 &= \int \left(\frac{1}{2} u_1^2 + u_0 E u_2 \right) d_{\hbar} x \\ H_2 &= \int \left(\frac{1}{3} u_1^3 + u_0 E u_0 E^2 u_3 + u_0 u_1 E u_2 + u_0 E u_1 E u_2 \right) d_{\hbar} x \\ &\vdots \end{aligned}$$

4 The continuous limit

The aim of this section is to consider the limit $\hbar \rightarrow 0$ of discrete systems (2.10). The class of this systems is determined by the choice of the algebra \mathcal{C} . Let us assume that the dynamical fields from \mathcal{C} depend on \hbar in such a way that the expansion, with respect to \hbar near zero, is of the form

$$u_i(x) = u_i^{(0)}(x) + u_i^{(1)}(x)\hbar + O(\hbar^2),$$

i.e. u_i in the limit $\hbar \rightarrow 0$ tends to $u_i^{(0)}$. In further considerations instead of $u_i^{(0)}$ we will still use u_i . In the continuous limit \mathcal{C} becomes the algebra, denoted by \mathcal{C}_0 , of polynomial functions in $u_i(x)$:

$$\mathcal{C}_0 \ni f(\mathbf{u}) := \sum_{m \geq 0} \sum_{i_1, \dots, i_m \geq 0} a^{i_1 i_2 \dots i_m} u_{i_1}(x) u_{i_2}(x) \cdots u_{i_m}(x).$$

In general, the limit of discrete systems (2.10) does not have to exist. To take the limit, one should first expand the coefficients of $\mathbf{K}(\mathbf{u})$ into a Taylor series with respect to \hbar near 0, i.e.

$$E^m u = e^{m\hbar \mathcal{X} \partial_x} u = u + m\hbar \mathcal{X} u_x + \frac{m^2}{2} \hbar^2 (\mathcal{X} \mathcal{X}_x u_x + \mathcal{X}^2 u_{2x}) + O(\hbar^3).$$

Thus, the continuous limit of (2.10) exists only if in the above expansion zero order terms in \hbar will mutually cancel. In this case, as $\hbar \rightarrow 0$, the discrete systems (2.10) go to the systems of hydrodynamic type given in the following form

$$\mathbf{u}_t = \mathcal{X} \mathbf{A}(\mathbf{u}) \mathbf{u}_x, \quad (4.1)$$

where $\mathbf{A}(\mathbf{u})$ is the matrix with coefficients from \mathcal{C}_0 , and the continuous limit is indeed the dispersionless limit.

Proposition 4.1 *Assuming that fields $u_i(x)$ vanish as $|x| \rightarrow \infty$, in the continuous limit, the functionals from Definition 2.4 are given by*

$$\int (\cdot) d_0 x : \mathcal{C}_0 \rightarrow \mathbb{K} \quad f[\mathbf{u}] \mapsto F(\mathbf{u}) = \int f(\mathbf{u}) d_0 x = \int_{-\infty}^{\infty} f(\mathbf{u}(x)) \frac{dx}{\mathcal{X}(x)}. \quad (4.2)$$

Proof. For the continuous case (2.9) the proof is straightforward. In the case of discrete functionals (2.8), proceeding analogously as for Riemann integral, we have

$$\begin{aligned} \int f[\mathbf{u}] d_0 x &\equiv \lim_{\hbar \rightarrow 0} \int f[\mathbf{u}] d_{\hbar} x = \lim_{\hbar \rightarrow 0} \sum_{n \in \mathbb{Z}} \hbar f[\mathbf{u}(\sigma_{n\hbar}(x))] \\ &= \lim_{\hbar \rightarrow 0} \sum_{n \in \mathbb{Z}} f[\mathbf{u}(\sigma_{n\hbar}(x))] \left(\frac{\mu_{\hbar}(x)}{\hbar} \right)^{-1} \mu_{\hbar}(x) = \int_{-\infty}^{\infty} f(\mathbf{u}(x)) \frac{dx}{\mathcal{X}(x)}. \end{aligned}$$

□

Then, bi-vectors π are matrices with coefficients of the operator form $a\mathcal{X}\partial_x b$, where $a, b \in \mathcal{C}_0$. With respect to the duality map defined by the 'dispersionless' functional (4.2) the adjoint of the operator ∂_x is given as

$$(\partial_x)^\dagger = \frac{\mathcal{X}_x}{\mathcal{X}} - \partial_x. \quad (4.3)$$

Consequently, the variational derivatives of functionals $F = \int f d_0 x = \int_{-\infty}^{\infty} f \frac{dx}{\mathcal{X}}$ are given by the derivatives of densities f with respect to the fields u_i , i.e.

$$\frac{\delta F}{\delta u_i} = \frac{\partial f}{\partial u_i}.$$

Example 4.2 *The dispersionless limit of the system (3.6) together with its Hamiltonian structure with respect to the first Poisson tensor is given by*

$$(u_i)_{t_1} = \mathcal{X}[(u_{i+1})_x + i u_i (u_0)_x] = \pi_0^{ij} \frac{\delta H_1}{\delta u_j}, \quad (4.4)$$

where

$$\pi_0^{ij} = j \mathcal{X} \partial_x u_{i+j} + i u_{i+j} \mathcal{X} \partial_x \quad \text{and} \quad H_1 = \int \left(u_1 + \frac{1}{2} u_0^2 \right) d_0 x.$$

The Hamiltonian representation of the systems (4.1) with the functional (4.2) following directly from the continuous limit and leads to the nonstandard form with the adjoint operation for differential operator given by (4.3). A more natural representation is the one with the components $\mathcal{X}(x)$ included in the densities of functionals given in the standard form

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} \mathcal{X}(x)^{-1} f(\mathbf{u}(x)) dx \equiv \int_{-\infty}^{\infty} \varphi(\mathbf{u}(x)) dx,$$

for which the variational derivatives preserve the form $\frac{\delta F}{\delta u_i} = \frac{\partial \varphi}{\partial u_i}$. As a consequence, bi-vectors π from the previous representation must be multiplied on the right-hand side by \mathcal{X} . Now, the adjoint of the operator ∂_x takes the standard form $(\partial_x)^\dagger = -\partial_x$. Therefore, in what follows we will use only the natural Hamiltonian representation of dispersionless systems (4.1).

Example 4.3 *The natural Hamiltonian structure of (4.4) is given by*

$$\pi_0^{ij} = j \mathcal{X} \partial_x \mathcal{X} u_{i+j} + i u_{i+j} \mathcal{X} \partial_x \mathcal{X} \quad \text{and} \quad H_1 = \int_{-\infty}^{\infty} \mathcal{X}^{-1} \left(u_1 + \frac{1}{2} u_0^2 \right) dx.$$

In the next section we will consider the R -matrix formalism of the dispersionless systems (4.1), that can be considered as the continuous limit of the formalism presented in Section 3. That it is really the case would be clear in Section 6.

5 R -matrix approach to integrable dispersionless systems on \mathbb{R}

The theory of classical R -matrices on commutative algebras, with the multi-Hamiltonian formalism, was given in [17]. Here we follow the particular scheme of R -matrix parallel to the one developed in [18, 19].

Let us consider the algebra of polynomials in p with the finite highest order:

$$\mathcal{A} = \mathcal{A}_{\geq k-1} \oplus \mathcal{A}_{< k-1} = \left\{ \sum_{i \geq k-1}^N u_i(x) p^i \right\} \oplus \left\{ \sum_{i < k-1} u_i(x) p^i \right\} \quad (5.1)$$

and with the Lie structure induced by the Poisson bracket in the form

$$\{f, g\} := p\mathcal{X}(x) \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right) \quad f, g \in \mathcal{A}. \quad (5.2)$$

The subsets $\mathcal{A}_{\geq k-1}$ and $\mathcal{A}_{< k-1}$ of \mathcal{A} are Lie subalgebras only for $k = 1$ and $k = 2$. Thus, the classical R -matrices $R = P_{\geq k-1} - \frac{1}{2}$ determine the Lax hierarchies:

$$L_{t_n} = \{(L^n)_{\geq k-1}, L\} = \pi_0 dH_n = \pi_1 dH_{n-1} \quad n \in \mathbb{Z}_+ \quad k = 1, 2, \quad (5.3)$$

that are generated by powers of the Lax functions $L \in \mathcal{A}$ given in the form:

$$k = 1 : \quad L = p + u_0 + u_1 p^{-1} + u_2 p^{-2} + \dots = p + \sum_{i \geq 0} u_i p^{-i} \quad (5.4)$$

$$k = 2 : \quad L = u_0 p + u_1 + u_2 p^{-1} + u_3 p^{-2} + \dots = \sum_{i \geq 0} u_i p^{1-i}. \quad (5.5)$$

The first dispersionless chains from (5.3) for $k = 1$ take the following form

$$\begin{aligned} (u_i)_{t_1} &= \mathcal{X} [(u_{i+1})_x + i u_i (u_0)_x] \\ (u_i)_{t_2} &= 2\mathcal{X} [(u_{i+2})_x + u_0 (u_{i+1})_x + (i+1) u_{i+1} (u_0)_x + i u_i u_0 (u_0)_x + i u_i (u_1)_x] \\ &\vdots, \end{aligned} \quad (5.6)$$

and for $k = 2$

$$\begin{aligned} (u_i)_{t_1} &= \mathcal{X} [u_0 (u_{i+1})_x + i u_{i+1} (u_0)_x] \\ (u_i)_{t_2} &= 2\mathcal{X} [u_0^2 (u_{i+2})_x + (i+1) u_0 u_{i+2} (u_0)_x + u_0 u_1 (u_{i+1})_x + i u_{i+1} (u_0 u_1)_x] \\ &\vdots. \end{aligned} \quad (5.7)$$

Example 5.1 For $\mathcal{X} = 1$ the chains (5.6) and (5.7) are dispersionless Toda and modified Toda chains, respectively, while for $\mathcal{X} = x$ the chains (5.6) and (5.7) are dispersionless limits of the q -analogues of Toda and modified Toda.

The appropriate trace form is defined as

$$\text{Tr}(A) := \int_{-\infty}^{\infty} \mathcal{X}^{-1} \text{res}(A p^{-1}) dx,$$

where $\text{res}(A) := a_{-1}$ for $A = \sum_i a_i p^i$, and the inner product on \mathcal{A} is given by

$$(A, B) := \text{Tr}(AB).$$

Proposition 5.2 *The above inner product is nondegenerate, symmetric and ad-invariant with respect to the Poisson bracket, i.e.*

$$(\{A, B\}, C) = (A, \{B, C\}) \quad A, B, C \in \mathcal{A}.$$

Proof. The nondegeneracy and symmetricity is obvious. The ad-invariance is a consequence of the following equality: $\text{Tr} \{A, B\} = 0$, which is valid for arbitrary $A, B \in \mathcal{A}$. \square

Then, the differentials $dH(L)$ of functionals $H(L) \in \mathcal{F}(\mathcal{A})$ related to the Lax functions (5.4-5.5) have the form:

$$\begin{aligned} k = 1 : \quad dH &= \mathcal{X} \sum_{i \geq 0} \frac{\delta H}{\delta u_i} p^i \\ k = 2 : \quad dH &= \mathcal{X} \sum_{i \geq 0} \frac{\delta H}{\delta u_i} p^{i-1}. \end{aligned}$$

The bi-Hamiltonian structure of the Lax hierarchies (3.3) is defined through the compatible (for fixed k) Poisson tensors:

$$k = 1, 2 : \quad \pi_0 : dH \mapsto \{L, (dH)_{<k-1}\} + (\{dH, L\})_{<2-k}$$

and

$$\begin{aligned} k = 1 : \quad \pi_1 : dH &\mapsto \{L, (dHL)_{<0}\} + L(\{dH, L\})_{<1} + \{\partial_x^{-1} \text{res}(\mathcal{X}^{-1} \{dH, L\} p^{-1}), L\} \\ k = 2 : \quad \pi_1 : dH &\mapsto \{L, (dHL)_{<1}\} + L(\{dH, L\})_{<0}. \end{aligned}$$

Then, for Hamiltonians

$$H_n(L) = \frac{1}{n+1} \text{Tr}(L^{n+1}) \quad dH_n(L) = L^n,$$

the explicit bi-Hamiltonian structure of (3.3) is given by

$$(u_i)_{t_n} = \sum_{j \geq 0} \pi_0^{ij} \frac{\delta H_n}{\delta u_j} = \sum_{j \geq 0} \pi_1^{ij} \frac{\delta H_{n-1}}{\delta u_j} \quad i \geq 0.$$

So, the Poisson tensors for $k = 1$ are given by

$$\begin{aligned} \pi_0^{ij} &= \mathcal{X} [j \partial_x u_{i+j} + i u_{i+j} \partial_x] \mathcal{X} \\ \pi_1^{ij} &= \mathcal{X} \left[\sum_{k=0}^i ((j-k) u_k \partial_x u_{i+j-k} + (i-k) u_{i+j-k} \partial_x u_k) + i(j+1) u_i \partial_x u_j \right. \\ &\quad \left. + (j+1) \partial_x u_{i+j+1} + (i+1) u_{i+j+1} \partial_x \right] \mathcal{X} \end{aligned}$$

where the related Hamiltonians are

$$\begin{aligned} H_0 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} u_0 dx \\ H_1 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} \left(u_1 + \frac{1}{2} u_0^2 \right) dx \\ H_2 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} \left(u_2 + 2u_0 u_1 + \frac{1}{3} u_0^3 \right) dx \\ &\vdots \end{aligned}$$

For $k = 2$ we have the first Poisson tensor

$$\begin{aligned}\pi_0^{10} &= \mathcal{X} \partial_x \mathcal{X} u_0 & \pi_0^{01} &= u_0 \mathcal{X} \partial_x \mathcal{X} \\ \pi_0^{ij} &= \mathcal{X} [(j-1) \partial_x u_{i+j-1} + (i-1) u_{i+j-1} \partial_x] \mathcal{X} \quad i, j \geq 2,\end{aligned}$$

where all remaining π_0^{ij} are equal zero, and the second one

$$\pi_1^{ij} = \mathcal{X} \left[\sum_{k=0}^{i-1} ((j-k) u_k \partial_x u_{i+j-k} + (i-k) u_{i+j-k} \partial_x u_k) + (1-i) u_i \partial_x u_j \right] \mathcal{X}.$$

Finally

$$\begin{aligned}H_0 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} u_1 dx \\ H_1 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} \left(\frac{1}{2} u_1^2 + u_0 u_2 \right) dx \\ H_2 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1} \left(\frac{1}{3} u_1^3 + u_0^2 u_3 + 2 u_0 u_1 u_2 \right) dx \\ &\vdots\end{aligned}$$

One can observe that the chains, together with the bi-Hamiltonian structures, constructed in this section are dispersionless limits of the discrete chains considered in Section 3.

6 Deformation quantization procedure

The aim of this section is formulation of the inverse procedure to the dispersionless limit considered earlier. Using the formalism of quantization deformation (for the references see [15]) the unified approach to the lattice and field soliton systems was presented in [15]. Here we follow the scheme from that article.

The Poisson bracket (5.2) can be written in the form

$$\{f, g\} := f (p \partial_p \wedge \mathcal{X}(x) \partial_x) g \quad f, g \in \mathcal{A},$$

where the derivations $p \partial_p$ and $\mathcal{X}(x) \partial_x$ commute. Hence, it can be quantized in infinitely many ways via \star -products being deformed multiplications

$$f \star^\alpha g = f \exp \left[\frac{\hbar}{2} ((\alpha+1) p \partial_p \otimes \mathcal{X}(x) \partial_x + (\alpha-1) \mathcal{X}(x) \partial_x \otimes p \partial_p) \right]. \quad (6.1)$$

This \star -product for $\alpha = 0$ and $\alpha = 1$ is the generalization of the Moyal and Kuperschmidt-Manin products, respectively. Expanding (6.1) one finds that

$$f \star^\alpha g = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} \sum_{j=0}^k (\alpha+1)^{k-j} (\alpha-1)^j [(p \partial_p)^{k-j} (\mathcal{X} \partial_x)^j f] \cdot [(\mathcal{X} \partial_x)^{k-j} (p \partial_p)^j g]. \quad (6.2)$$

The algebra \mathcal{A} (5.1) with the multiplication defined as (6.1), with fixed α , is the associative, but not commutative, algebra with Lie bracket, being a deformed Poisson bracket, defined as

$$\{f, g\}_{\star^\alpha} = \frac{1}{\hbar} (f \star^\alpha g - g \star^\alpha f). \quad (6.3)$$

Then, in the limit $\hbar \rightarrow 0$, we have that

$$\begin{aligned}\lim_{\hbar \rightarrow 0} f \star^\alpha g &= fg \\ \lim_{\hbar \rightarrow 0} \{f, g\}_{\star^\alpha} &= \{f, g\}.\end{aligned}$$

The algebra \mathcal{A} with \star^α -product will be denoted as \mathcal{A}_α .

The associativity property of \star^α -products is a purely algebraic consequence of their construction. For the simple proof see [15]. Moreover, we could treat these products only formally not requiring a convergence of the sum in (6.2). In order to make the \star^α -products consistent with the introduced formalism of grain structures we assume that vector fields $\mathcal{X}\partial_x$ are such that the formula (2.4) is valid. From the simple observation:

$$(p\partial_p)^k p^m = m^k p^m,$$

one finds that

$$\begin{aligned}p^m \star^\alpha u(x) &= \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} (\alpha + 1)^k m^k (\mathcal{X}\partial_x)^k u(x) p^m = e^{m(\alpha+1)\frac{\hbar}{2}\mathcal{X}\partial_x} u(x) p^m = E^{m\frac{\alpha+1}{2}} u(x) p^m \\ u(x) \star^\alpha p^m &= \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} (\alpha - 1)^k m^k (\mathcal{X}\partial_x)^k u(x) p^m = e^{m(\alpha-1)\frac{\hbar}{2}\mathcal{X}\partial_x} u(x) p^m = E^{m\frac{\alpha-1}{2}} u(x) p^m,\end{aligned}$$

where the last equalities follow from the above assumption and (2.5).

It is important that the decomposition of (5.1) into Lie subalgebras after deformation quantization is preserved and they are still Lie subalgebras with respect to the Lie bracket (6.3). Hence, we have Lax hierarchies

$$L_{t_n} = \{(L^n)_{\geq k-1}, L\}_{\star^\alpha} \quad n \in \mathbb{Z}_+ \quad k = 1, 2, \quad (6.4)$$

which be still well-defined for Lax functions in the form (5.4-5.5). Notice that now, the Lax hierarchies are generated by powers with respect to \star^α -products, i.e. $L^n = L \star^\alpha \dots \star^\alpha L$. The first chains from Lax hierarchies (6.4) are

$$\begin{aligned}k = 1 : \quad (u_i)_{t_1} &= \frac{1}{\hbar} \left[(E - 1) E^{\frac{\alpha-1}{2}} u_{i+1} + u_i (1 - E^{-i}) E^{i\frac{1-\alpha}{2}} u_0 \right] \\ k = 2 : \quad (u_i)_{t_1} &= \frac{1}{\hbar} \left[E^{i\frac{1-\alpha}{2}} u_0 E^{\frac{\alpha+1}{2}} u_{i+1} - E^{\frac{\alpha-1}{2}} u_{i+1} E^{-i(\frac{\alpha+1}{2})} u_0 \right].\end{aligned}$$

One can observe that they coincide with the respective discrete systems from Section 3 for $\alpha = 1$.

Nevertheless, all algebras \mathcal{A}_α are gauge equivalent under the isomorphism

$$D^{\alpha'-\alpha} : \mathcal{A}_\alpha \rightarrow \mathcal{A}_{\alpha'} \quad D^{\alpha'-\alpha} = \exp \left[(\alpha - \alpha') \frac{\hbar}{2} \mathcal{X}(x) \partial_x p \partial_p \right],$$

such that

$$\begin{aligned}f \star^{\alpha'} g &= D^{\alpha'-\alpha} \left[D^{\alpha-\alpha'} f \star^\alpha D^{\alpha-\alpha'} g \right] \\ \{f, g\}_{\star^{\alpha'}} &= D^{\alpha'-\alpha} \left\{ D^{\alpha-\alpha'} f, D^{\alpha-\alpha'} g \right\}_{\star^\alpha}.\end{aligned}$$

It is also straightforward to prove that under the above isomorphism the Lax hierarchy structure is preserved. Let $L_\alpha = \sum_i u_i p^i \in \mathcal{A}_\alpha$ and $L_{\alpha'} = \sum_i u'_i p^i \in \mathcal{A}_{\alpha'}$. Then, the transformation between fields follows

$$L_{\alpha'} = D^{\alpha' - \alpha} L_\alpha \quad \Rightarrow \quad u'_i = E^{i \frac{\alpha - \alpha'}{2}} u_i.$$

On the other hand, directly from (6.1) the following commutation rules result:

$$\begin{aligned} u \star v &= uv \\ p^m \star p^n &= p^{m+n} \\ p^m \star u &= (e^{m\hbar\mathcal{X}\partial_x} u) \star p^m = E^m u \star p^m \\ u \star p^m &= p^m \star (e^{-m\hbar\mathcal{X}\partial_x} u) = p^m \star E^{-m} u, \end{aligned}$$

being independent of the choice of \star^α -product, therefore we have skipped the related index. Hence, we can quantize separately the algebra \mathcal{A} to the following algebra [23]

$$\mathfrak{a} = \left\{ \sum_i u_i \star p^i \right\},$$

which obviously is associative under the above commutation rules. Notice that the algebra \mathfrak{a} differs from algebras \mathfrak{g}_α as in \mathfrak{a} we also deformed the polynomial functions, i.e. we are not using the standard multiplication any more. Let us point out that the algebra \mathfrak{a} is trivially equivalent to the algebra \mathcal{A}_1 as $u \star^1 p^m = up^m$ and $p^m \star^1 u = E^m up^m$. Also, it is straightforward to see that \mathfrak{a} is isomorphic to the algebra of shift operators \mathfrak{g} (3.1) defined on the grain structure by some discrete one-parameter group of diffeomorphisms on \mathbb{R} . Hence, it is clear that the algebra (5.1) with Poisson bracket (5.2) is the limit, $\hbar \rightarrow 0$, of the algebra (3.1) of shift operators with the Lie structure defined by the commutator.

7 Conclusions

In the present article we have introduced a general framework of integrable discrete systems on \mathbb{R} . This formalism is based on the construction of shift operators by means of discrete one-parameter groups of diffeomorphisms on \mathbb{R} that are determined by infinitesimal generators $\mathcal{X}\partial_x$. Particularly, if $\mathcal{X} = 1$ or $\mathcal{X} = x$ the discrete systems considered are of lattice Toda or q -deformed Toda type, respectively. Although the construction of integrable systems related to different $\mathcal{X}\partial_x$ is completely parallel, they, in general, are not equivalent, similarly as the vector fields on \mathbb{R} are not equivalent.

The two vector fields $\mathcal{X}(x)\partial_x$, $\mathcal{X}'(x')\partial_{x'}$ and the related discrete systems are equivalent if integrating

$$\int \frac{dx}{\mathcal{X}(x)} = \int \frac{dx'}{\mathcal{X}'(x')}$$

one finds a bijective map between x and x' , otherwise related one-parameter groups of diffeomorphisms do not transform into one another.

Consider the vector fields from Example 2.3. Let $\mathcal{X}(x) = x^{1-n}$ for $n \neq 0$ and $\mathcal{X}'(x') = 1$ (the lattice case). Then one finds that $x' = \frac{1}{n}x^n$, which is a bijection for n being odd. Hence, all the discrete systems generated by $\mathcal{X}\partial_x = x^{1-n}\partial_x$ with odd n can be reduced to the original

lattice Toda type systems. For $n = 0$ $\mathcal{X}(x) = x$ (the q -discrete case) and let again $\mathcal{X}'(x') = 1$. Then $x = e^{x'}$ is not the bijection. However, if the domain of dynamical fields of q -discrete systems is restricted to $x \in \mathbb{R}_+$, then the above map is a bijection and q -discrete systems on \mathbb{R}_+ became equivalent to the lattice systems on \mathbb{R} .

Acknowledgement

The work of M. Błaszak, M. Gürses and B. Silindir was partially supported by the Turkish Academy of Sciences and by the Scientific and Technological Research Council of Turkey. Moreover M. Błaszak and B. Szablikowski were supported by MNiI research grant no. N202.

References

- [1] C. Kassel, *Cyclic homology of differential operators, the Virasoro algebra and a q -analogue*, Comm. Math. Phys. **146** (1992) 343-356
- [2] D.H. Zhang, *Quantum deformation of KdV hierarchies and their infinitely many conservation laws*, J. Phys. A: Math. Gen. **26** (1993) 2389-2407
- [3] J. Mas and M. Seco, *The algebra of q -pseudodifferential symbols and the $q - W_{\text{KP}}^{(n)}$ algebra*, J. Math. Phys. **37** (1996) 6510-6529
- [4] E. Frenkel, *Deformations of the KdV hierarchy and related soliton equations*, Int. Math. Res. Not. **2** (1996) 55-76
- [5] B. Khesin, V. Lyubashenko and C. Roger, *Extensions and contractions of the Lie algebra of q -pseudodifferential symbols on the circle*, J. Funct. Anal. **143** (1997) 55-97
- [6] M. Adler, E. Horozov and P. van Moerbeke, *The solution to the q -KdV equation*, Phys. Lett. **A 242** (1998) 139-151
- [7] K. Takasaki, *q -Analogue of Modified KP Hierarchy and its Quasi-Classical Limit*, Lett. Math. Phys. **70** (2005) 165-181
- [8] J. He, Y. Li and Y. Cheng, *q -Deformed KP Hierarchy and q -Deformed Constrained KP Hierarchy*, SIGMA **2** (2006) Paper 060
- [9] A. Dimakis and F. Müller-Hoissen, *Functional representations of integrable hierarchies*, arXiv:nlin.SI/0603018
- [10] M. A. Semenov-Tian-Shansky, *What is a classical r -matrix?*, Funct. Anal. Appl. **17** (1983) 259
- [11] B. G. Konopelchenko and W. Oevel, *An r -matrix approach to nonstandard classes of integrable equations*, Publ. RIMS, Kyoto Univ. **29** (1993) 581-666
- [12] M. Błaszak and K. Marciniak, *R -matrix approach to lattice integrable systems*, J. Math. Phys. **35** (1994) 4661
- [13] W. Oevel, *Poisson Brackets in Integrable Lattice Systems in Algebraic Aspects of Integrable Systems* edited by A.S. Fokas and I.M. Gelfand, Progress in Nonlinear Differential Equations Vol. 26 (Birkhäuser-Boston) (1996) 261
- [14] M. Błaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Texts and Monographs in Physics (Springer-Verlag, Berlin, 1998) 350 pp.
- [15] M. Błaszak and B. M. Szablikowski, *From dispersionless to soliton systems via Weyl-Moyal-like deformations*, J. Phys A: Math. Gen. **36** (2003) 12181-12203
- [16] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. **7** (1995) 743-808

- [17] Li Luen-Chau, *Classical r -Matrices and Compatible Poisson Structures for Lax Equations in Poisson Algebras*, Commun. Math. Phys. **203** (1999) 573-592
- [18] M. Błaszak and B. M. Szablikowski, *Classical R -matrix theory of dispersionless systems: I. $(1+1)$ -dimension theory*, J. Phys A: Math. Gen. **35** (2002) 10325-10344
- [19] B. M. Szablikowski and M. Błaszak, *Meromorphic Lax representations of $(1+1)$ -dimensional multi-Hamiltonian dispersionless systems*, J. Math. Phys. **47** (2006) 092701
- [20] B. Aulbach, S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, Nonlinear dynamics and quantum dynamical systems (Gaussig, 1990) Math. Res. 59 (Akademie-Verlag, Berlin, 1990), pp. 9-20
- [21] M. Gürses, G. Sh. Guseinov and B. Silindir, *Integrable equations on time scales*, J. Math. Phys **46** (2005) 113510
- [22] G. Carlet, B. Dubrovin and Y. Zhang, *The Extended Toda Hierarchy*, Mosc. Math. J. **4** (2004) 313-332
- [23] A. Das and Z. Popowicz, *Properties of Moyal-Lax representation*, Phys. Lett. B **510** (2001) 264-270